

On infinite dimensional Volterra type operators

Farrukh Mukhamedov¹, Mansoor Saburov²

February 26, 2009

In this paper we study Volterra type operators on infinite dimensional simplex. It is provided a sufficient condition for Volterra type operators to be bijective. Furthermore it is proved that the condition is not necessary.

Mathematics Subject Classification: 15A51, 47H60, 46T05, 92B99.

Key words: Infinite dimensional simplex, Volterra type operators, cubic stochastic operators.

1 Introduction

Since Lotka and Volterra's seminal and pioneering works (see [10]) many decades ago, modeling of interacting, competing species have received considerable attention in the fields of biology, ecology, mathematics (for example, see [3, 9]). In their remarkably simple deterministic model, Lotka and Volterra considered two coupled nonlinear differential equations that mimic the temporal evolution of a two-species system of competing predator and prey populations. They demonstrated that coexistence of both species was not only possible but inevitable in their model. Moreover, similar to observations in real populations, both predator and prey densities in this deterministic system display regular oscillations in time, with both the amplitude and the period determined by the prescribed initial conditions. Note that in [1, 2, 4, 5] finite dimensional Volterra and more general quadratic operators were studied.

When a system is large enough, it is interesting to investigate quadratic Volterra operators define on an infinite dimensional simplex. First studies in this direction were considered in [6]. Iterations of such operators define more complicated nonlinear operators. To better understanding the dynamics of such operators, it is important to study such nonlinear operators. The aim of this paper is to study more general class of nonlinear operators which contains a particular case of that quadratic Volterra operators. It is provided a sufficient condition for Volterra type operators to be bijective

¹Department of Comput. & Theor. Sci., Faculty of Sciences, IIUM, P.O. Box, 141, 25710, Kuantan, Pahang, Malaysia, *e-mail:* far75m@yandex.ru

²Department of Mechanics & Mathematics, National University of Uzbekistan, Vuzgorodok, 100174 Tashkent, Uzbekistan

2 Volterra type operators

In this section we give definition of Volterra type operators and study its basic properties. We prove that such operators are bijective under certain condition.

Let

$$\ell_1 = \{x = (x_n) : \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty\}$$

be the space of all absolutely convergent sequences. The set

$$S = \{x = (x_n) \in \ell_1 : x_i \geq 0, \sum_{n=1}^{\infty} x_n = 1\}$$

is called *an infinite dimensional simplex*.

It is known [7] that $S = \overline{\text{convh}(\text{Extr}(S))}$ form a convex closed set, here

$$\text{Extr}(S) = \{e^{(n)} = (\underbrace{0, 0, \dots, 0}_n, 1, 0, \dots)\}$$

is the extremal points of the S and $\text{conv}(A)$ is the convex hull of a set A .

Let $\alpha \subset N$ be an arbitrary set. The set

$$S_\alpha = \{x \in S : x_k = 0, \forall k \in N \setminus \alpha\}$$

is called a *face* of the simplex. It is clear that S_α is also will be the simplex.

A *relatively interior* of a face S_α is defined by

$$\text{ri}S_\alpha = \{x \in S_\alpha : x_k > 0, \forall k \in \alpha\}.$$

In what follows we are interested in the following operator $V : S \rightarrow \ell_1$ defined by

$$(Vx)_k = x_k(1 + f_k(x)), \quad k \in N, \quad x \in S, \quad (1)$$

where a mapping $\mathbf{f} : x \in S \rightarrow (f_1(x), f_2(x), \dots, f_n(x), \dots) \in \ell_1$.

One can see that

$$\left| \sum_{k=1}^{\infty} x_k f_k(x) \right| \leq \sum_{k=1}^{\infty} |x_k f_k(x)| \leq \sum_{k=1}^{\infty} |f_k(x)| < \infty$$

for any $x \in S$, that means that the operator V is well defined.

Theorem 2.1 *Let an operator V be given defined by (1). The following conditions are equivalent:*

(i) *The operator V is continuous in ℓ_1 and $V(S) \subset S$. Moreover, $V(\text{ri}S_\alpha) \subset \text{ri}S_\alpha$ for every $\alpha \subset N$.*

(ii) *The mapping \mathbf{f} satisfies the following conditions:*

1^0 \mathbf{f} *is continuous on ℓ_1 topology.*

2⁰ For every $k \in N$ one holds $f_k(x) \geq -1$ for all $x \in S$.

3⁰ For any $x \in S$ one has $\sum_{k=1}^{\infty} x_k f_k(x) = 0$.

4⁰ For any $\alpha \subset N$ one has $f_k(x) > -1$ for all $x \in riS_\alpha$ and $k \in \alpha$.

Proof (i) \Rightarrow (ii). The continuity of V implies 1⁰. Take $x \in S$, then $V(x) \in S$ yields that

$$(a) \quad (V(x))_k \geq 0,$$

$$(b) \quad \sum_{k=1}^{\infty} (V(x))_k = 1.$$

Hence, from (a) it follows that $x_k(1 + f_k(x)) \geq 0$, which implies 2⁰. From (b) one has

$$\sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} x_k f_k(x) = 1,$$

which immediately yields 3⁰.

Let $x \in riS_\alpha$, then $V(x) \in riS_\alpha$, which with (1) and $x_k > 0$ for all $k \in \alpha$ implies that $f_k(x) > -1$ for all $k \in \alpha$.

The implication (ii) \Rightarrow (i) is evident.

We say that an operator $V : S \rightarrow S$ defined by (1) is *Volterra type* if one of the conditions of Theorem 2.1 is satisfied. The corresponding mapping \mathbf{f} is called *generating* for V . By \mathcal{V} we denote the set of all Volterra type operators.

From Theorem 2.1 we have

Corollary 2.2 *The set \mathcal{V} is convex, and for any $V_1, V_2 \in \mathcal{V}$ one has $V_1 \circ V_2 \in \mathcal{V}$, here \circ means composition of operators.*

Recall that the point $x \in S$ is called *fixed point* of V , if $Vx = x$. The set of all fixed points of V is denoted by $Fix(V)$. From Theorem 2.1 we immediately get the following

Corollary 2.3 *For any Volterra type operator V one holds*

$$(i) \quad Extr(S) \subset Fix(V);$$

$$(ii) \quad \text{Restriction of } V \text{ to any face of the simplex is again Volterra type operator.}$$

Let us consider more particular case of a mapping \mathbf{f} . Namely one has the following

Theorem 2.4 *Let $f_k : \ell_1 \rightarrow R$ ($k \in N$) be linear functionals. Then a mapping $\mathbf{f} : S \rightarrow \ell_1$ defined by $\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_n(x), \dots)$ satisfies the conditions 1⁰ – 4⁰ iff one has*

$$f_k(x) = \sum_{i=1}^{\infty} a_{ki} x_i, \quad k \in N \tag{2}$$

with

$$a_{ki} = -a_{ik}, \quad |a_{ki}| \leq 1 \quad \text{for every } k, i \in N$$

Note that Volterra type operators with generating mappings given by (2) are called *quadratic Volterra operators*. In [6] such quadratic Volterra operators have been studied, and it was shown that any such kind of operator is a bijection of S . In the case under consideration, i.e. for Volterra type operators we could not state an analogous result.

Example 2.1 *Let us consider 1-dimensional simplex, i.e. $S^1 = [0, 1]$. Define a Volterra type operator $V : [0, 1] \rightarrow [0, 1]$ by*

$$V(x) = x(1 - \sin \pi x), \quad x \in [0, 1].$$

A direct inspection shows that the defined operator is not bijection.

Now we are interested in finding some sufficient conditions for Volterra type operators to be bijective.

Theorem 2.5 *Let V be a Volterra type operator given by (1). Let*

$$\sum_{k=1}^{\infty} x_k f_k(y) + \sum_{k=1}^{\infty} y_k f_k(x) \leq 0 \quad \text{for every } x, y \in S. \quad (3)$$

be satisfied. Then V is a bijection of S

Proof. First let us prove that the V is an injection. Suppose that there are two distinct elements x, y such that

$$V(x) = V(y). \quad (4)$$

Without loss of generality we may assume that $x_i > 0, y_i > 0, \forall i \in N$. If it is not true, then there is a face S_α , for some subset $\alpha \subset N$ of S such that $x, y \in riS_\alpha$ that is $x_i > 0, y_i > 0, \forall i \in \alpha$. According to Theorem 2.1 we have $V(S_\alpha) \subset S_\alpha$ therefore, due to Corollary 2.3 we can restrict V to S_α which is Volterra type too.

Now from (4) one gets

$$x_k(1 + f_k(x)) = y_k(1 + f_k(y)), \quad \forall k \in N$$

or

$$x_k - y_k + x_k f_k(x) - y_k f_k(x) = y_k f_k(y) - y_k f_k(x), \quad \forall k \in N$$

$$(x_k - y_k)(1 + f_k(x)) = -y_k(f_k(x) - f_k(y)), \quad \forall k \in N.$$

Since $1 + f_k(x) > 0, \forall x \in riS$ and $y_k > 0, \forall k \in N$, then

$$\text{sgn}(x_k - y_k) = -\text{sgn}(f_k(x) - f_k(y)), \quad \forall k \in N. \quad (5)$$

Hence

$$(x_k - y_k)(f_k(x) - f_k(y)) \leq 0, \quad \forall k \in N \quad (6)$$

whence

$$\sum_{k=1}^{\infty} (x_k - y_k)(f_k(x) - f_k(y)) \leq 0. \quad (7)$$

Note that the last series absolutely converges, since

$$\begin{aligned} \left| \sum_{k=1}^{\infty} (x_k - y_k)(f_k(x) - f_k(y)) \right| &\leq \sum_{k=1}^{\infty} |x_k - y_k| |f_k(x) - f_k(y)| \\ &\leq \sum_{k=1}^{\infty} (x_k + y_k) (|f_k(x)| + |f_k(y)|) \\ &\leq 2 \left(\sum_{k=1}^{\infty} |f_k(x)| + \sum_{k=1}^{\infty} |f_k(y)| \right) < \infty \end{aligned}$$

From (7) one finds

$$\sum_{k=1}^{\infty} x_k f_k(x) - \sum_{k=1}^{\infty} x_k f_k(y) - \sum_{k=1}^{\infty} y_k f_k(x) + \sum_{k=1}^{\infty} y_k f_k(y) \leq 0. \quad (8)$$

By means of condition 3⁰, (8) can be rewritten by

$$\sum_{k=1}^{\infty} x_k f_k(y) + \sum_{k=1}^{\infty} y_k f_k(x) \geq 0. \quad (9)$$

From (3) and (9) we conclude that (6) is true if and only if

$$(x_k - y_k)(f_k(x) - f_k(y)) = 0, \quad \forall k \in N.$$

The equality (5) with the last equality implies that $x = y$. Thus, $V : S \rightarrow S$ is injective.

Now let us show that V is onto. Denote

$$\begin{aligned} A_1 &= \{[1, n] \subset N : n \in N\}, \\ A_2 &= \{a \subset [1, n] : |[1, n] \setminus a| \geq 2, n \in N\}, \\ A_3 &= \{b \subset N : a \subset b, a \in A_1 \cup A_2, |N \setminus b| < \infty\}, \\ A &= A_1 \cup A_2 \cup A_3. \end{aligned}$$

In A define an order by inclusion, i.e. $a \leq b$ means that $a \subset b$ for $a, b \in A$. It is evident that A is a completely ordering set. We will prove surjection of V by using transfer induction method with respect to the set A . Clearly that operator V is a surjection on $S_{\{1\}}$. Suppose that for an element $a \in A$ the operator V is a surjective on S_b for every $b < a$. Let us show that the V is a surjection on S_a as well. Assume that $V(S_a) \neq S_a$. For the boundary ∂S_a of S_a we have $\partial S_a = \bigcup_{c \in A: c < a} S_c$. According to the assumption of the induction we get

$$V(\partial S_a) = \partial S_a. \quad (10)$$

On the other hand, there exist $x, y \in riS_a$ such that $x \in V(S_a)$, $y \notin V(S_a)$. The segment $[x, y]$ contains at least one boundary point z of the set $V(S_a)$. Since $V : S_a \rightarrow V(S_a)$ is continuous and bijection, then the boundary point goes to boundary one. Therefore, for $z \in riS_a$, $V^{-1}(z) \in \partial S_a$, which contradicts to (10).

Remark 2.1 Note that the functionals (2) described in Theorem 2.4 satisfy the condition (3).

3 Cubic Volterra operators

From the previous section one arises a question: is there non-trivial (except for linear functionals f_k) examples of Volterra type operators for which condition (3) is satisfied. In this section we introduce so called cubic Volterra operators, and provide an examples of such operators which satisfy condition (3). Moreover, we establish that that condition is indeed sufficient, i.e. an example of bijective cubic Volterra operator will be given, which does not satisfy (3).

Recall that a mapping $V : S \rightarrow S$ is called *cubic stochastic operator* (shortly *c.s.o.*) if it is defined by

$$(Vx)_k = \sum_{i,j,l=1}^{\infty} p_{ijl,k} x_i x_j x_l, \quad \forall k \in N \quad (11)$$

for all $x \in S$, here

$$\sum_{k=1}^{\infty} p_{ijl,k} = 1, \quad \text{and} \quad p_{ijl,k} = p_{\pi(i)\pi(j)\pi(l)} \geq 0, \quad (12)$$

where π is any permutation of the index set $\{i, j, l\}$.

Note that cubic stochastic operators were studied in [3],[8].

We say that a c.s.o. V is called *cubic Volterra operator* (*c.V.o.*) if any face of the simplex is invariant with respect to V . One can prove the following

Theorem 3.1 A c.s.o. V is cubic Volterra operator if and only if $p_{ijl,k} = 0$ whenever $k \notin \{i, j, l\}$. Moreover, any cubic Volterra operator is Volterra type, and it can be represented by

$$(Vx)_k = x_k \left(x_k^2 + 3x_k \sum_{i \in N_k} p_{ikk,k} x_i + 3 \sum_{i \in N_k} p_{iik,k} x_i^2 + 6 \sum_{i,j \in N_k, i < j} p_{ijk,k} x_i x_j \right), \quad (13)$$

for all $k \in N$, where $N_k = N \setminus \{k\}$.

From this Theorem we immediately get the following

Corollary 3.2 If for a c.V.o. V the coefficients $\{p_{ijl,k}\}$ do not depend one of indexes $\{i, j, l\}$, then V becomes a quadratic Volterra operator.

Example 3.1. Let us consider c.V.o. $V_0 : S \rightarrow S$ defined by

7

Lemma 3.3 *Let*

$$W_k(x) = \left(\sum_{i=k}^{\infty} x_i \right)^3 + 3 \sum_{i=1}^{k-1} x_i \left(\sum_{i=k}^{\infty} x_i \right)^2 + 3 \sum_{i,j=1, i \leq j}^{k-1} x_i x_j \sum_{i=k}^{\infty} x_i, \quad k \in N.$$

Then for all $k \in N$ one has

$$W_k(x) = (Vx)_k + W_{k+1}(x).$$

Proof

$$\begin{aligned} W_k(x) &= \left(x_k + \sum_{i=k+1}^{\infty} x_i \right)^3 + 3 \sum_{i=1}^{k-1} x_i \left(x_k + \sum_{i=k+1}^{\infty} x_i \right)^2 \\ &\quad + 3 \sum_{i,j=1, i \leq j}^{k-1} x_i x_j \left(x_k + \sum_{i=k+1}^{\infty} x_i \right) \\ &= x_k^3 + 3x_k^2 \sum_{i=1}^{k-1} x_i + 3x_k \sum_{i=1}^{k-1} x_i \sum_{i=k+1}^{\infty} x_i + 3x_k \sum_{i,j=1, i \leq j}^{k-1} x_i x_j \\ &\quad + \left(\sum_{i=k+1}^{\infty} x_i \right)^3 + 3 \left(x_k + \sum_{i=1}^{k-1} x_i \right) \left(\sum_{i=k+1}^{\infty} x_i \right)^2 \\ &\quad + 3 \left(\sum_{i,j=1, i \leq j}^{k-1} x_i x_j + x_k \sum_{i=1}^{k-1} x_i + x_k^2 \right) \sum_{i=k+1}^{\infty} x_i \\ &= x_k \left(x_k^2 + 3 \sum_{i=1}^{k-1} x_i \sum_{i=k}^{\infty} x_i + 3 \sum_{i,j=1, i \leq j}^{k-1} x_i x_j \right) + W_{k+1}(x) \\ &= (Vx)_k + W_{k+1}(x). \end{aligned}$$

From Lemma 3.3 we obtain

$$1 = \left(\sum_{i=1}^{\infty} x_i \right)^3 = (Vx)_1 + W_2(x) = (Vx)_1 + (Vx)_2 + W_3(x) = \cdots = \sum_{i=1}^{\infty} (Vx)_i$$

Now we show that the operator (15) is bijective. To this end, first we prove that the operator is injective. Indeed, let $x, y \in S$ and $x \neq y$. Then there exists $k_0 \in N$ such that

$$x_{k_0} \neq y_{k_0}, \quad x_i = y_i, \quad \forall i = \overline{1, k_0 - 1}.$$

From (15) we find that

$$(Vx)_i = (Vy)_i, \quad \forall i = \overline{1, k_0 - 1}.$$

From (16) and $x_i = y_i \forall i = \overline{1, k_0 - 1}$ it follows that

$$C := \sum_{i=1}^{k_0-1} x_i - \sum_{i,j=1, i < j}^{k_0-1} x_i x_j = \sum_{i=1}^{k_0-1} y_i - \sum_{i,j=1, i < j}^{k_0-1} y_i y_j \geq 0.$$

Consider a function $g(t) = t^3 + 3C \cdot t$, which is strictly increasing on the segment $[0, 1]$. Therefore, for $x_{k_0} \neq y_{k_0}$ we get

$$(Vx)_{k_0} = g(x_{k_0}) \neq g(y_{k_0}) = (Vy)_{k_0},$$

which means $Vx \neq Vy$.

Since the restriction of the operator (15) to any face of S is again of type (15), then by similar argument used in the proof of Theorem 2.5, one can establish that the operator (15) is surjective.

Let us show that (3) is not satisfied. From (15) one sees that

$$f_k(x) = x_k^2 + 3 \sum_{i=1}^{k-1} x_i - 3 \sum_{i,j=1, i < j}^{k-1} x_i x_j - 1.$$

The inequality (3) is not satisfied if we put $x = e^{(1)}, y = e^{(2)}$, indeed

$$\sum_{k=1}^{\infty} e_k^{(1)} f_k(e^{(2)}) + \sum_{k=1}^{\infty} e_k^{(2)} f_k(e^{(1)}) = -1 + 2 = 1 > 0.$$

Acknowledgement

The F.M. thanks Research Endowment Grant B of International Islamic University Malaysia, SAGA Fund P77c by MOSTI through the Academy of Sciences Malaysia(ASM) and TUBITAK.

References

- [1] Bernstein S.N. The solution of a mathematical problem concerning the theory of heredity. *Ucheniye-Zapiski N.-I. Kaf.Ukr.Otd.Mat.*, **1**(1924), 83–115 (Russian) .
- [2] Ganikhodzhaev.R.N. Quadratic stochastic operators, Lyapunov functions and tournaments. *Russian Acad.Sci. Sbornik.Math.*, **76**(1993), 489–506.
- [3] Hofbauer J., Sigmund K., *Evolutionary Games and Population Dynamics*, Cambridge University Press, Cambridge, 1998.
- [4] Kesten H. Quadratic transformations: a model for population growth. I,II *Adv.Appl.Prob.*, **2**(1970), 1–82; 179–228

- [5] Lyubich Yu.I. *Mathematical structures in population genetics*, Springer-Verlag, Berlin, 1992.
- [6] Mukhamedov, F. Akin H., Temir S. On infinite dimensional quadratic Volterra operators, *Jour. Math. Anal. Appl.* **310**(2005), 533–556.
- [7] Roy N. Extreme points and $l_1(\Gamma)$ – spaces. *Proc. Amer. Math. Soc.* **86** (1982), 216–218.
- [8] Rozikov U A. Hamraev A.Yu. On Cubic operators defined on finite dimensional simplex. *Ukr.Mat. Jour.*, 56(2004), 1424-1433.
- [9] Takeuchi Y., *Global dynamical properties of Lotka–Volterra systems*, World Scientific, 1996.
- [10] Volterra V., Lois de fluctuation de la population de plusieurs espèces coexistant dans le même milieu, *Association Franc. Lyon* **1926** (1927), 96–98.